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GENERALIZED CAUCHY–JENSEN (m, n)-ADDITIVE MAPPINGS IN RANDOM NORMED SPACES

John Michael Rassias and Hark-Mahn Kim†

Abstract. In the paper we consider a general Cauchy–Jensen (m, n)-additive functional equation and establish new theorems about the generalized Hyers–Ulam stability of the approximate Cauchy–Jensen (m, n)-additive mappings in random normed spaces.

1. Introduction

One of the interesting questions in the theory of functional analysis concerning the stability problem of functional equations is as follows: when is it true that a mapping satisfying approximately a functional equation must be close to an exact solution of the given functional equation? The first stability problem was raised by S.M. Ulam [25] during his talk at the University of Wisconsin in 1940 as follows: Let G be a group and G' a metric group with metric $\rho(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$. For very general functional equations, the concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? In the next year 1941, D.H. Hyers [13] was the first one who presented affirmatively the result concerning the stability of functional equations for approximately linear mappings $f : E \to E'$ between Banach spaces. A generalized version of the Hyers theorem

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* This work was supported by research fund of Chungnam National University.
† Correspondence should be addressed to Hark-Mahn Kim, hmkim@cnu.ac.kr.

for approximate additive mappings which allows the Cauchy difference to be unbounded was given by T. Aoki [3] and D.G. Bourgin [5]. In 1978, Th.M. Rassias [22] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate linear mappings. P. Găvruta [11] obtained generalized result of Th.M. Rassias' theorem which allow the Cauchy difference to be controlled by a general function. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [10, 14, 23]. These stability results can be applied in stochastic analysis [16], financial and actuarial mathematics [9], as well as in psychology and sociology [1, 2].

Before taking up the subject, we recall that a function $F : \mathbb{R} \to [0,1]$ is called a distribution function if it is nondecreasing and left continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$. The class of all distribution functions F with $F(0) = 0$ is denoted by D_+. For any $a \geq 0$, ε_a is the element of D_+ defined by

$$\varepsilon_a(t) = \begin{cases} 0, & \text{if } \ t \leq a, \\ 1, & \text{if } \ t > a. \end{cases}$$

Then the maximal element for D_+ partially ordered by the usual ordering of functions is the distribution function ε_0. Throughout this paper, we shall use the terminology, notations and conventions of theory of probabilistic metric spaces and random normed spaces [7, 12, 17, 24]. The following definition can be found in the reference [12, 24].

Definition 1.1. Let X be a real linear spaces, $F : X \to D_+$ be a mapping denoted by F_x for $x \in X$ and T be a t-norm. The triple (X, F, T) is called a random normed space (briefly, RN-space) if the following conditions are satisfied:

- (RN1) $F_x = \varepsilon_0$ iff $x = 0$, the null vector;
- (RN2) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$, and $x \in X$;
- (RN3) $F_{x+y}(t_1 + t_2) = T(F_x(t_1), F_x(t_2))$ for all $x, y \in X$ and $t_1, t_2 > 0$.

Here $T : [0,1] \times [0,1] \to [0,1]$ is a continuous binary operation that is associative, commutative, nondecreasing and has 1 as identity. The three typical t-norms are $Prod(a,b) := ab$, $W(a,b) := \max\{a + b - 1, 0\}$ and $T_M(a,b) := \min\{a,b\}$.

A sequence $\{x_n\}$ in an RN-space (X, F, T) converges to $x \in X$ if $\lim_{n\to\infty} F_{x_n - x}(t) = 1, \forall t > 0$. We remark that if a sequence $\{x_n\}$ converges to x in an RN-space (X, F, T), then $\lim_{n\to\infty} F_{x_n}(t) = F_x(t)$ [24].

A sequence $\{x_n\}$ in an RN-space (X,F,T) is called a Cauchy sequence if $\lim_{m,n\to\infty} F_{x_n-x_m}(t)=1, \forall t>0$. The RN-space (X,F,T) is said to be complete if every Cauchy sequence in X is convergent. During the last two decades, the stability problems of various functional equations in random normed spaces have been investigated by a number of mathematicians and there are several applicable interesting results concerning these stability problems; see [6, 8, 15, 21] and references therein.

Now, we consider a mapping $f: X\to Y$ between linear spaces satisfying the following functional equation

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\ne i_j,\forall j\in\{1,\cdots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m}+\sum_{l=1}^{n-m} x_{k_l}\right) = \frac{n-m+1}{n}\binom{n}{m}\sum_{i=1}^n f(x_i) \tag{1.1}$$

for all $x_1,\cdots,x_n\in X$, where $n,m\in\mathbb{N}$ are fixed integers with $n\ge 2$, $1\le m\le n$. Recently, the authors [19, 18, 20] have investigated approximate Cauchy–Jensen (m,n)-additive mappings in quasi-β-normed spaces, and in C^*-algebras, respectively, associated with stability theorems of the equation (1.1). In the sequel, we establish investigate the generalized Hyers–Ulam stability problem for the general Cauchy–Jensen (m,n)-additive functional equation (1.1) with $n\ge 2$ in random normed spaces in the present paper.

2. Approximate Cauchy–Jensen (m,n)-additive mappings

Let $L:=n-m+1>1$ be a fixed positive integer with $n\ge 2$ and let $1\le m<n$ otherwise specific reference. For notational convenience, given a mapping $f: X\to Y$, we define the difference operator $Df: X^n\to Y$ of the equation (1.1) by

$$\begin{aligned} &Df(x_1,x_2,\cdots,x_n)\\ &:= \sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\ne i_j,\forall j\in\{1,\cdots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m}+\sum_{l=1}^{n-m} x_{k_l}\right) - \frac{n-m+1}{n}\binom{n}{m}\sum_{i=1}^n f(x_i)\end{aligned}$$

for all n-variables $x_1, \cdots, x_n \in X, (n \geq 2)$ which acts as a perturbation of the equation (1.1). Throughout this section, we assume that X is a real linear space, $(Y, F, T_M = \min)$ is a complete RN-space and $(Z, F', T'_M = \min)$ is an RN-space. Now we are going to investigate the modified Hyers–Ulam stability of the functional equation (1.1).

LEMMA 2.1. [4] *Let $(X, F, T_M = \min)$ be a RN-space. Define $E_{\lambda,F} : X \to [0, \infty)$ by*

$$E_{\lambda,F}(x) = \inf\{t > 0 : F_x(t) > 1 - \lambda\}$$

for each $\lambda \in (0, 1)$ and $x \in X$. Then we have

$$E_{\lambda,F}(x_1 - x_n) \leq \sum_{i=2}^{n} E_{\lambda,F}(x_{i-1} - x_i)$$

for all $x_1, \ldots, x_n \in X$. Further, a sequence $\{x_n\}$ converges to x in $(X, F, T_M = \min)$ if and only if $E_{\lambda,F}(x_n - x) \to 0$, and the sequence $\{x_n\}$ is a Cauchy if and only if $E_{\lambda,F}(x_n - x_m) \to 0$.

THEOREM 2.2. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$F_{Df(x_1,x_2,\cdots,x_n)}(t) \geq F'_{\varphi(x_1,\cdots,x_n)}(t) \tag{2.1}$$

and $\varphi : X^n \to Z$ is a mapping for which there is a constant $l \in \mathbb{R}$ satisfying $0 < |l| < L$ such that

$$F'_{\varphi(Lx_1,\cdots,Lx_n)}(t) \geq F'_{l\varphi(x_1,\cdots,x_n)}(t) \tag{2.2}$$

for all n-variables $x_1, \cdots, x_n \in X$, and $t > 0$. Then we can find a unique Cauchy–Jensen (m, n)-additive mapping $A : X \to Y$ satisfying the equation (1.1) and the approximate inequality

$$\begin{aligned} E_{\lambda,F}(f(x) - A(x)) \quad &\leq \quad \frac{E_{\lambda,F'}(\varphi(x, \cdots, x))}{\binom{n}{m}(L - |l|)}, \\ \text{i.e.,} \quad F_{f(x)-A(x)}(t) \quad &\geq \quad F'_{\varphi(x,\cdots,x)}\left(\binom{n}{m}(L - |l|)t\right), \quad t > 0 \end{aligned} \tag{2.3}$$

for all $x \in X$.

Proof. We observe from (2.2) that

$$\begin{aligned} E_{\lambda,F'}(\varphi(Lx_1,\cdots,Lx_n)) &= \inf\{t>0 : F'_{\varphi(Lx_1,\cdots,Lx_n)}(t) > 1-\lambda\} \\ &\le \inf\{t>0 : F'_{l\varphi(x_1,\cdots,x_n)}(t) > 1-\lambda\} \\ &= \inf\{t>0 : F'_{\varphi(x_1,\cdots,x_n)}(\frac{t}{|l|}) > 1-\lambda\} \qquad (2.4) \\ &= \inf\{|l|t>0 : F'_{\varphi(x_1,\cdots,x_n)}(t) > 1-\lambda\} \\ &= |l|E_{\lambda,F'}(\varphi(x_1,\cdots,x_n)) \end{aligned}$$

for all $x_1,\cdots,x_n \in X$, $t>0$, and $\lambda \in (0,1)$. It follows from (2.1) that

$$\begin{aligned} E_{\lambda,F}(Df(x_1,\cdots,x_n)) &= \inf\{t>0 : F_{Df(x_1,\cdots,x_n)}(t) > 1-\lambda\} \\ &\le \inf\{t>0 : F'_{\varphi(x_1,\cdots,x_n)}(t) > 1-\lambda\} \qquad (2.5) \\ &= E_{\lambda,F'}(\varphi(x_1,\cdots,x_n)) \end{aligned}$$

for all $x_1,\cdots,x_n \in X$, and $t>0$. Now, substituting x for $x_1,\cdots,x_n$ in the functional inequality (2.5), we obtain

$$\begin{aligned} E_{\lambda,F}\left(\binom{n}{m}f(Lx) - \binom{n}{m}Lf(x)\right) &\le E_{\lambda,F'}(\varphi(x,\cdots,x)), \\ \text{or,} \quad E_{\lambda,F}\left(\frac{f(Lx)}{L} - f(x)\right) &\le \frac{1}{\binom{n}{m}L}E_{\lambda,F'}(\varphi(x,\cdots,x)) \qquad (2.6) \end{aligned}$$

for all $x \in X$. Therefore it follows from (2.4), (2.6) with $L^i x$ in place of x, and Lemma 2.1 that

$$\begin{aligned} E_{\lambda,F}\left(\frac{f(L^s x)}{L^s} - \frac{f(L^{k+s}x)}{L^{k+s}}\right) &\le \sum_{i=s}^{k+s-1} E_{\lambda,F}\left(\frac{f(L^i x)}{L^i} - \frac{f(L^{i+1}x)}{L^{i+1}}\right) \\ &\le \sum_{i=s}^{k+s-1} \frac{1}{\binom{n}{m}L^{i+1}}E_{\lambda,F'}(\varphi(L^i x,\cdots,L^i x)) \\ &\le \sum_{i=s}^{k+s-1} \frac{|l|^i}{\binom{n}{m}L^{i+1}}E_{\lambda,F'}(\varphi(x,\cdots,x)) \qquad (2.7) \\ &= \frac{E_{\lambda,F'}(\varphi(x,\cdots,x))}{\binom{n}{m}L}\sum_{i=s}^{k+s-1}(\frac{|l|}{L})^i \end{aligned}$$

for all $x \in X$ and any integers $k>0, s\ge 0$. Thus it follows by taking the limit $s\to\infty$ in (2.7) that a sequence $\left\{\frac{f(L^k x)}{L^k}\right\}$ is Cauchy in the complete space (Y, F, T_M) and so it converges in Y. Therefore we see that a mapping $A : X \to Y$ defined by

$$A(x) := \lim_{k\to\infty} \frac{f(L^k x)}{L^k} = \lim_{k\to\infty} \frac{f((n-m+1)^k x)}{(n-m+1)^k}$$

is well defined for all $x \in X$. This means that

$$\begin{aligned} \lim_{k\to\infty} E_{\lambda,F}\left(\frac{f(L^k x)}{L^k} - A(x)\right) = 0, \quad &\Leftrightarrow \quad \lim_{k\to\infty} F_{\frac{f(L^k x)}{L^k} - A(x)}(t) = 1, \\ &\Leftrightarrow \quad \lim_{k\to\infty} F_{\frac{f(L^k x)}{L^k}}(t) = F_{A(x)}(t) \end{aligned}$$

for all $t > 0$. In addition it is clear from (2.5) that the following inequality

$$\begin{aligned} E_{\lambda,F}\left(\frac{Df(L^k x_1, \cdots, L^k x_n)}{L^k}\right) \quad &\le \quad E_{\lambda,F'}\left(\frac{\varphi(L^k x_1, \cdots, L^k x_n)}{L^k}\right) \\ &\le \quad E_{\lambda,F'}\left(\frac{|l|^k}{L^k}\varphi(x_1, \cdots, x_n)\right) \\ &\le \quad \frac{|l|^k}{L^k} E_{\lambda,F'}\left(\varphi(x_1, \cdots, x_n)\right) \\ &\to \quad \text{as } k \to \infty \end{aligned}$$

holds for all $x_1, \cdots, x_n \in X$. Therefore we obtain

$$\lim_{k\to\infty} F_{\frac{Df(L^k x_1, \cdots, L^k x_n)}{L^k}}(t) = \lim_{k\to\infty} F_{DA(x_1,\cdots,x_n)}(t) = 1, \ \forall t > 0$$

which implies $DA(x_1, \cdots, x_n) = 0$ by (RN1). Hence the mapping A is Cauchy–Jensen (m, n)-additive.

Now, taking the limit $k \to \infty$ in (2.7) with $s = 0$, we see that

$$\begin{aligned} &E_{\lambda,F}(f(x) - A(x)) \\ &\quad \le E_{\lambda,F}\left(f(x) - \frac{f(L^k x)}{L^k}\right) + E_{\lambda,F}\left(\frac{f(L^k x)}{L^k} - A(x)\right) \\ &\quad \le \frac{E_{\lambda,F'}(\varphi(x, \cdots, x))}{\binom{n}{m} L} \sum_{i=0}^{k-1} (\frac{|l|}{L})^i + E_{\lambda,F}\left(\frac{f(L^k x)}{L^k} - A(x)\right) \\ &\quad \le \frac{1}{\binom{n}{m}(L - |l|)} E_{\lambda,F'}(\varphi(x, \cdots, x)), \end{aligned}$$

that is,

$$\begin{aligned} & \inf\{t>0: F_{f(x)-A(x)}(t) > 1-\lambda\} \\ \leq & \frac{1}{\binom{n}{m}(L-|l|)} \inf\{t>0: F'_{\varphi(x,\cdots,x)}(t) > 1-\lambda\} \\ = & \inf\{t>0: F'_{\varphi(x,\cdots,x)}\left(\binom{n}{m}(L-|l|)t\right) > 1-\lambda\}, \end{aligned}$$

which yields

$$F_{f(x)-A(x)}(t) \quad \geq \quad F'_{\varphi(x,\cdots,x)}\left(\binom{n}{m}(L-|l|)t\right), \ t>0$$

for all $x \in X$. Thus we find that A is a Cauchy additive mapping satisfying the inequality (2.3) near the approximate mapping $f : X \to Y$.

To prove the afore-mentioned uniqueness, we assume now that there is another Cauchy–Jensen (m,n)-additive mapping $A' : X \to Y$ which satisfies the inequality (2.3). Then one establishes by the last equality and (2.3) that

$$\begin{aligned} F_{A(x)-A'(x)}(t) &= \lim_{k\to\infty} F_{\frac{f(L^k x)}{L^k} - \frac{A'(L^k x)}{L^k}}(t) \\ &\geq \lim_{k\to\infty} F'_{\varphi(L^k x,\cdots,L^k x)}\left(\binom{n}{m}(L-|l|)L^k t\right) \\ &\geq \lim_{k\to\infty} F'_{|l|^k\varphi(x,\cdots,x)}\left(\binom{n}{m}(L-|l|)L^k t\right) \\ &\geq \lim_{k\to\infty} F'_{\varphi(x,\cdots,x)}\left(\binom{n}{m}(L-|l|)\frac{L^k}{|l|^k} t\right) \\ &= 1, \quad t>0, \end{aligned}$$

because $F'_{\varphi(x,\cdots,x)} \in D_+$, and $\sup_{t\in\mathbb{R}} F'_{\varphi(x,\cdots,x)}(t) = 1$. Therefore one obtains $A(x) - A'(x) = 0$ for all $x \in X$, which completes the proof. □

Theorem 2.3. *Assume that a mapping $f : X \to Y$ satisfies the inequality (2.1) and φ is a mapping for which there is a constant $l \in \mathbb{R}$ satisfying $|l| > L$ such that*

$$F'_{\varphi(\frac{x_1}{L},\cdots,\frac{x_n}{L})}(t) \geq F'_{\varphi(x_1,\cdots,x_n)}(|l|t) \tag{2.8}$$

for all n-variables $x_1, \cdots, x_n \in X$, and $t > 0$. Then there exists exactly a Cauchy–Jensen (m,n)-additive mapping $A : X \to Y$ satisfying the *equation (1.1) and the inequality*

$$E_{\lambda,F}(f(x)-A(x)) \leq \frac{E_{\lambda,F'}(\varphi(x,\cdots,x))}{\binom{n}{m}(|l|-L)}$$

$$\textit{i.e.,}\quad F_{f(x)-A(x)}(t) \geq F'_{\varphi(x,\cdots,x)}\left(\binom{n}{m}(|l|-L)t\right),\ t>0$$

for all $x \in X$.

Proof. It follows from (2.8) and (2.5) that

$$E_{\lambda,F'}\left(\varphi(\frac{x_1}{L},\cdots,\frac{x_n}{L})\right) \leq \frac{1}{|l|}E_{\lambda,F'}(\varphi(x_1,\cdots,x_n)),$$

$$E_{\lambda,F}\left(f(x)-Lf(\frac{x}{L})\right) \leq \frac{1}{\binom{n}{m}}E_{\lambda,F'}(\varphi(\frac{x}{L},\cdots,\frac{x}{L})) \tag{2.9}$$

for all $x \in X$. Therefore it follows from (2.9) with $L^{-i}x$ in place of x, and Lemma 2.1 that

$$\begin{aligned} &E_{\lambda,F}\left(L^s f(\frac{x}{L^s}) - L^{k+s}f(\frac{x}{L^{k+s}})\right) \\ &\leq \sum_{i=s}^{k+s-1} E_{\lambda,F}\left(L^i f(\frac{x}{L^i}) - L^{i+1}f(\frac{x}{L^{i+1}})\right) \\ &\leq \sum_{i=s}^{k+s-1} \frac{L^i}{\binom{n}{m}} E_{\lambda,F'}(\varphi(\frac{x}{L^{i+1}},\cdots,\frac{x}{L^{i+1}})) \\ &\leq \sum_{i=s}^{k+s-1} \frac{L^i}{\binom{n}{m}|l|^{i+1}} E_{\lambda,F'}(\varphi(x,\cdots,x)) \\ &= \frac{E_{\lambda,F'}(\varphi(x,\cdots,x))}{\binom{n}{m}|l|} \sum_{i=s}^{k+s-1} (\frac{L}{|l|})^i \end{aligned}$$

for all $x \in X$ and any integers $k > 0, s \geq 0$.

The remaining assertion goes through by the similar way to corresponding part of Theorem 2.2. □

We obtain the following corollary concerning the stability for approximate Cauchy–Jensen (m,n)-additive mappings of which difference operator $Df : X^n \to Y$ is uniformly bounded by a constant.

Corollary 2.4. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$F_{Df(x_1,x_2,\cdots,x_n)}(t) \geq F'_{\varepsilon}(t)$$

for all n-variables $x_1, \cdots, x_n \in X$, and for some constant $\varepsilon \geq 0$. Then there exists a unique Cauchy–Jensen (m, n)-additive mapping $A : X \to Y$ satisfying the equation

$$DA(x_1, x_2, \cdots, x_n) = 0$$

and the inequality

$$\begin{aligned} E_{\lambda,F}(f(x) - A(x)) &\leq \frac{E_{\lambda,F'}(\varepsilon)}{\binom{n}{m}(n-m)} \\ \text{i.e.,} \quad F_{f(x)-A(x)}(t) &\geq F'_{\varepsilon}\left(\binom{n}{m}(n-m)t\right), \ t > 0 \end{aligned}$$

for all $x \in X$.

We remark that if $\varepsilon = 0$, then $F_{Df(x_1,x_2,\cdots,x_n)}(t) \geq F'_{\varepsilon}(t) = 1$, and so $Df(x_1, x_2, \cdots, x_n) = 0$. Thus we get $f = A$ because $E_{\lambda,F'}(0) = 0$.

Now, in the next theorem we are to consider a singular case $m = n$ of Theorem 2.2 and Theorem 2.3 concerning the stability of the equation (1.1).

Theorem 2.5. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (2.1) and φ is a mapping for which there is a constant $l \in \mathbb{R}$ satisfying $0 < |l| < n$ such that*

$$F'_{\varphi(nx_1,\cdots,nx_n)}(t) \geq F'_{l\varphi(x_1,\cdots,x_n)}(t)$$

for all n-variables $x_1, \cdots, x_n \in X$, and $t > 0$. Then we can find a unique Cauchy–Jensen (m, n)-additive mapping $A : X \to Y$ satisfying the equation (1.1) and the inequality

$$\begin{aligned} E_{\lambda,F}(f(x) - A(x)) &\leq \frac{E_{\lambda,F'}(\varphi(0, \cdots, \overbrace{x}^{j-th}, 0 \cdots, 0))}{(1/|l| - 1/n)} \\ \text{i.e.,} \quad F_{f(x)-A(x)}(t) &\geq F'_{\varphi(0,\cdots,\underbrace{x}_{j-th},0\cdots,0)}\left((\frac{1}{|l|} - \frac{1}{n})t\right), \quad t > 0 \end{aligned}$$

for all $x \in X$ and all $j \in \{1, \cdots, n\}$.

Proof. For each $j = 1, \cdots, n$, substituting x for x_j and 0 for all x_i with $i \neq j$ in the functional inequality (2.5), one obtains

$$E_{\lambda,F}\left(f(\frac{x}{n}) - \frac{1}{n}f(x)\right) \leq E_{\lambda,F'}(\varphi(0, \cdots, \overbrace{x}^{j-th}, 0 \cdots, 0)),$$

so in addition,

$$E_{\lambda,F}\left(\frac{f(n^s x)}{n^s}-\frac{f(n^{k+s}x)}{n^{k+s}}\right)\le\sum_{i=s}^{k+s-1}E_{\lambda,F}\left(\frac{f(n^i x)}{n^i}-\frac{f(n^{i+1}x)}{n^{i+1}}\right)$$

$$\le\sum_{i=s}^{k+s-1}\frac{|l|^{i+1}}{n^i}E_{\lambda,F'}(\varphi(0,\cdots,\overbrace{x}^{j-th},0\cdots,0))$$

$$=\frac{E_{\lambda,F'}(\varphi(0,\cdots,\overbrace{x}^{j-th},0\cdots,0))}{(1/|l|-1/n)}$$

for all $x\in X$ and any integers $k>0, s\ge 0$.

The remaining assertion goes through by the similar way to corresponding part of Theorem 2.2. □

THEOREM 2.6. *Assume that a mapping $f:X\to Y$ with $f(0)=0$ satisfies the inequality (2.1) and φ is a mapping for which there is a constant $l\in\mathbb{R}$ satisfying $|l|>n$ such that*

$$F'_{\varphi(\frac{x_1}{n},\cdots,\frac{x_n}{n})}(t)\ge F'_{\varphi(x_1,\cdots,x_n)}(|l|t)$$

for all n-variables $x_1,\cdots,x_n\in X$, and $t>0$. Then there exists a unique Cauchy–Jensen (m,n)-additive mapping $A:X\to Y$ satisfying the equation (1.1) and the inequality

$$E_{\lambda,F}(f(x)-A(x))\le\frac{E_{\lambda,F'}(\varphi(0,\cdots,\overbrace{x}^{j-th},0\cdots,0))}{(1/n-1/|l|)}$$

$$\text{i.e.,}\quad F_{f(x)-A(x)}(t)\ge F'_{\varphi(0,\cdots,\underbrace{x}_{j-th},0\cdots,0)}\left((\frac{1}{n}-\frac{1}{|l|})t\right),\ t>0$$

for all $x\in X$ and all $j\in\{1,\cdots,n\}$.

Proof. The proof goes through by the similar way to corresponding part of Theorem 2.5. □

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John Michael Rassias
Pedagogical Department E.E.
National and Capodistrian University of Athens
Athens, 15342, Greece
E-mail: jrassias@primedu.uoa.gr

Hark-Mahn Kim
Department of Mathematics
Chungnam National University
Daejeon 34134, Korea
E-mail: hmkim@cnu.ac.kr